PHYSICAL REVIEW E VOLUME 57, NUMBER 2 FEBRUARY 1998

Chaos in effective classical and quantum dynamics

Lapo Casetti,* Raoul Gatto,† and Michele Modugno‡
Département de Physique Théorique, Université de Genève, 24 Quai Ernest-Ansermet, CH-1211 Genève, Switzerland

(Received 7 July 1997)

We investigate the dynamics of classical and quantum N-component ϕ^4 oscillators in the presence of an external field. In the large N limit the effective dynamics is described by two-degree-of-freedom classical Hamiltonian systems. In the classical model we observe chaotic orbits for any value of the external field, while in the quantum case chaos is strongly suppressed. A simple explanation of this behavior is found in the change in the structure of the orbits induced by quantum corrections. Consistently with Heisenberg's principle, quantum fluctuations are forced away from zero, removing in the effective quantum dynamics a hyperbolic fixed point that is a major source of chaos in the classical model. [S1063-651X(98)50302-9]

PACS number(s): 05.45.+b, 03.65.Sq, 11.15.Pg

The study of the quantum mechanics of those systems whose classical counterpart exhibits chaotic dynamics has attracted a lot of interest in recent years, and is an open and rapidly evolving field [1,2]. Chaos does not exist in the linear evolution of the quantum state vector, hence different approaches to identify quantum features that correspond to classical chaos have been developed, ranging from the application of random matrix theory to the statistical analysis of energy spectra [3] and to various semiclassical approximations [4]. Dynamical chaos in the actual quantum evolution may show up in mean-field approaches [5], or using Bohm's formulation of quantum mechanics to define quantum trajectories and quantum Lyapunov exponents [6]. Moreover, at a semiclassical level, the dynamics of quantum expectation values can be chaotic [7–9].

In this Rapid Communication we consider the effective dynamics of quantum expectation values as obtained in the large N limit. The purpose of the present work is twofold. First, we want to compare the quantum effective dynamics of a model system with the classical effective dynamics of the same system at the same level of approximation, in order to unambiguously detect the effect of the quantum corrections on dynamical chaos. This effect turns out to be a strong suppression of chaos with respect to the classical case. Second, we want to show that in our model the suppression of chaos in the quantum dynamics has a clear physical origin in the fact that quantum fluctuations must be nonvanishing, i.e., in the Heisenberg principle.

As a model system we consider a N-component ϕ^4 field theory in $d+1$ space-time dimensions, in the presence of an external field B. We shall limit ourselves to the case $d=0$, which allows us to offer a simple intuitive explanation of the effectiveness of quantum corrections in suppressing chaos. The Lagrangian that we consider is ($\alpha=1,\ldots,N$)

$$\mathcal{L}=\frac{1}{2}\,\dot{\phi}^{\alpha}\dot{\phi}_{\alpha}+\frac{1}{2}\mu^2\phi^{\alpha}\phi_{\alpha}-\frac{\lambda}{8N}(\phi^{\alpha}\phi_{\alpha})^2+B\frac{1}{N}\sum_{\alpha}\phi_{\alpha}. \tag{1}$$

We perform a $1/N$ expansion, keeping only the leading order term in both the classical and quantum case. In the former case we start by writing

$$\phi_{\alpha}=1/N\sum_{\beta}\phi_{\beta}+\delta\phi_{\alpha}\equiv\varphi+\delta\phi_{\alpha} \tag{2}$$

and we approximate the quadratic fluctuations by considering all of them equivalent in the large N limit,

$$\delta\phi_{\alpha}^2\simeq\xi^2. \tag{3}$$

By inserting Eqs. (2) and (3) into Eq. (1) it turns out that the dynamics of the mean field φ and of its root mean square (rms) fluctuation ξ is governed by the following effective Hamiltonian:

$$\mathcal{H}=1/2\,(\pi^2+\eta^2)+\lambda/8\,(\varphi^2+\xi^2-v_0^2)^2-B\varphi, \tag{4}$$

where the two canonically conjugated pairs of variables are φ,π and ξ,η, and $v_0\equiv\sqrt{2\mu^2/\lambda}$ is the minimum of the potential energy in Eq. (1) for $B=0$.

In the quantum case we consider the time evolution of the expectation value $\phi\equiv\Sigma_{\alpha}\langle\phi_{\alpha}\rangle/N$ from a given initial quantum state. This initial value quantum problem can be formulated by using the "closed time path" functional formalism [10]. The application of this formalism to the present case was developed by Cooper *et al.* [11], who showed that the evolution equations in the large N limit are (classical) Hamilton's equations for the effective Hamiltonian (we keep the dependence on the external source B),

$$\mathcal{H}=\frac{1}{2}(\pi^2+\eta^2)+\frac{\lambda}{8}(\varphi^2+\xi^2-v_0^2)^2+\frac{\hbar^2\sigma^2}{8\xi^2}-B\varphi, \tag{5}$$

where $\sigma\equiv 2n+1$, $n=\langle a^{\dagger}a\rangle$ being the expectation value, in the initial state, of the particle number operator for a single oscillator, and ξ the expectation value of the rms fluctuation of the fields, in close analogy to the classical case. In Ref. [11] it was also shown that $\mathcal{H}$ in Eq. (5) is just the expectation value of the full quantum Hamiltonian in a general mixed (initial) state characterized by a Gaussian density matrix.

*Present address: INFM, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy. Electronic address: lapo@polito.it
†Electronic address: gatto@sc2a.unige.ch
‡On leave from Dipartimento di Fisica, Università di Firenze, and INFN, sezione di Firenze, Largo Enrico Fermi 2, I-50125 Firenze, Italy. Electronic address: modugno@fi.infn.it

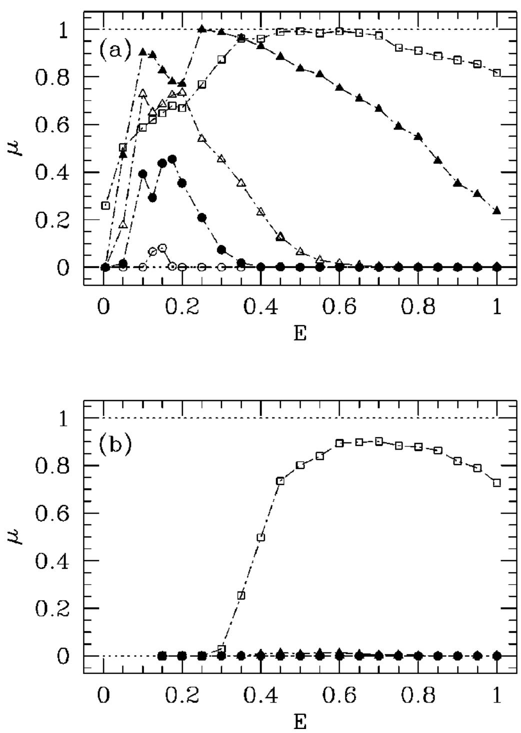

FIG. 1. Relative measure μ of the chaotic component of phase space vs energy E at different values of the external field B. Each point is an average over a sample of 1000 randomly chosen orbits. (a) Classical case; (b) quantum case. Symbols in both cases: $B=0.01$ (circles), $B=0.05$ (solid circles), $B=0.1$ (triangles), $B=0.3$ (solid triangles), $B=0.5$ (squares). Error bars are of the same size as the data points.

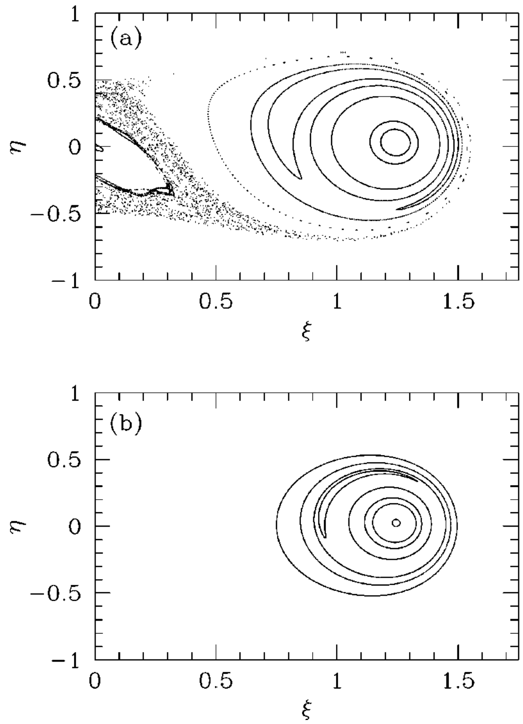

FIG. 2. Poincaré section of the Hamiltonian flow defined by Eq. (5) with $B=0.05$. (a) Classical case, (b) quantum case; the energy is $E=0.25$ in both cases.

We notice that at this order of the expansion the correspondence between the classical and the quantum cases is very strict, since both are described in terms of corresponding degrees of freedom, the mean field φ and its rms fluctuations ξ, and the classical effective Hamiltonian is just the quantum one with $\hbar=0$. This shows that the approximation (3) is equivalent to retaining only the classical contribution to the fluctuations. The quantum correction to the Hamiltonian (5) keeps the fluctuation ξ away from zero, consistently with Heisenberg's uncertainty principle. In the following we will always refer to the Hamiltonian (5), distinguishing between the classical and the quantum case according to the value of $\hbar$.

In the case of a vanishing external field $(B=0)$ the Hamiltonian (5) is integrable in both the classical $(\hbar=0)$ and quantum case $(\hbar\neq 0)$. The integrals of motion are the total energy E and the function

$$I=(\varphi\eta-\xi\pi)^2+(\hbar^2\sigma^2/4\xi^2)\,\varphi^2. \tag{6}$$

The conservation of I in Eq. (6) is due to the fact that the Hamiltonian (5) can be seen as the Hamiltonian, in cylindrical coordinates, of a particle in three dimensions moving in a central potential, φ being the azimuthal coordinate and ξ the radial one. The term $\hbar^2\sigma^2/4\xi^2\varphi^2$ is indeed the centrifugal barrier term [11], which shows up passing from Cartesian to cylindrical variables. From conservation of angular momentum it follows that $I\equiv L^2-L_z^2$ is conserved.

Switching on the external field $(B\neq 0)$, the two systems become nonintegrable. They have two integrable limits. These limits are: harmonic oscillators as $E\to 0$; and the integrable Hamiltonian with $B=0$ as $E\to\infty$. In fact, as the perturbation is linear in φ, it will become negligible with respect to the other terms in $\mathcal{H}$ for sufficiently large E. Hence, we expect that chaotic orbits may show up in an intermediate energy range, whose width will depend on B. We have studied the dynamics by numerically integrating the canonical equations of motion derived from the Hamiltonian (5) using a bilateral symplectic algorithm [12]. The values of the parameters were fixed by working in natural units $\hbar=1$ (in the quantum case) and by choosing $\lambda=v_0=\sigma=1$. The choice $\sigma=1$ corresponds to an initial vacuum state for the number operator $a^\dagger a$ [11].

We have studied chaos from both a qualitative and a quantitative point of view, i.e., we have calculated Poincaré sections [13] in the plane (ξ,η) and measured the Lyapunov exponent [14] of every single orbit. This has allowed us to also obtain an estimate of the relative measure μ of the chaotic regions in the phase space, defined as the ratio of the number of trajectories whose Lyapunov exponent is positive to the total number of trajectories, so that $0\leqslant\mu\leqslant 1$. For each value of the energy E and of the field B we have estimated $\mu(E,B)$ from a sample of 1000 orbits picked up at random in the allowed region on the section surface. We have considered the energy range $0\leqslant E\leqslant 1$, always fixing the zero of $E=0$ as the minimum energy allowed in the classical $B=0$ case, and field intensity range $0\leqslant B\leqslant 0.5$.

In Fig. 1 we plot $\mu(E,B)$ for the classical and the quantum case. It is evident that, as soon as $B\neq 0$, chaotic orbits suddenly appear in the classical case. For small values of B, i.e., $B=0.01$, such orbits are present only in a small interval of energies centered around $E\simeq 0.1$, then the chaotic energy interval broadens as B grows, and eventually fills the whole explored energy range as B becomes larger than 0.3. In the

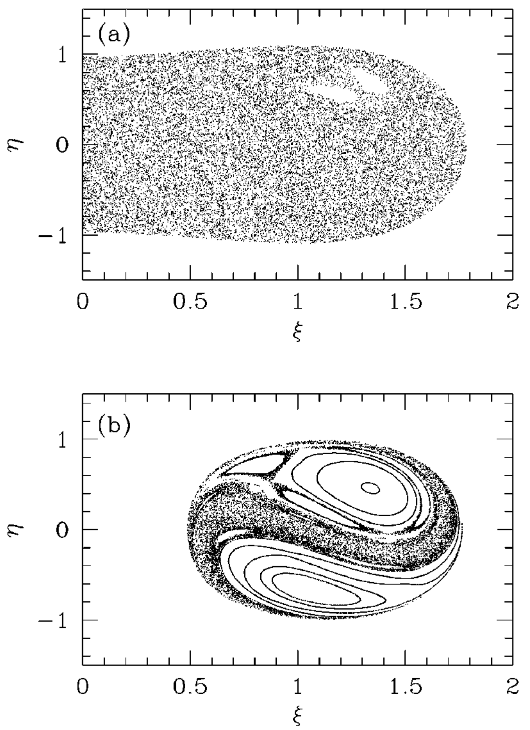

FIG. 3. The same as in Fig. 2 with $B=0.5$ and $E=0.6$.

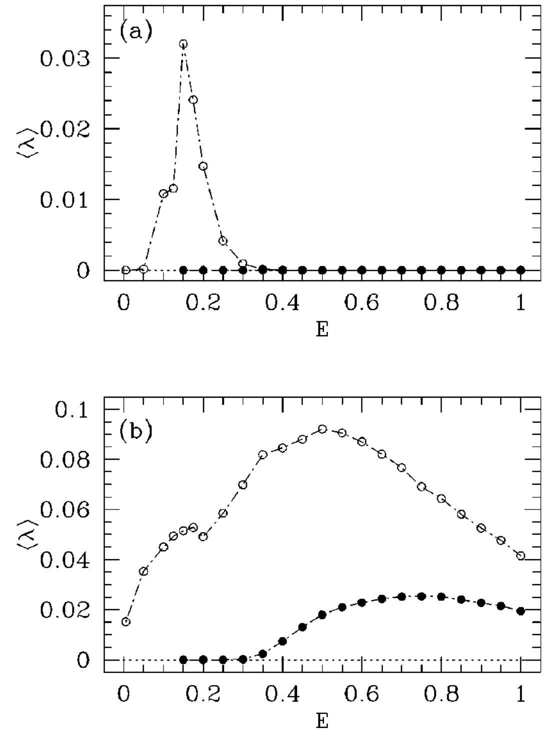

FIG. 4. Comparison between the values of $\langle\lambda\rangle(E)$ in the classical (open circles) and quantum (solid circles) case at (a) $B=0.05$ and (b) $B=0.5$.

quantum case the situation is completely different: no chaotic orbits are detected for $B\leqslant 0.3$, then chaos appears but at considerably larger values of E as compared to the classical case. To give an example, in Figs. 2 and 3 we show a comparison between Poincaré sections in the classical and the quantum case at two different values of the external field [notice that the external field B does not affect the shape of the region of the plane (ξ,η) accessible to the system, since B is coupled to φ].

In order to quantify the degree of chaos at a given energy E and external field B we have considered also the ensemble average $\langle\lambda\rangle(E,B)$ of the Lyapunov exponent over the samples of 1000 trajectories used to compute $\mu(E,B)$. A comparison between the classical and the quantum case is reported in Fig. 4 for the same values of B as in Figs. 2 and 3. Looking at these figures it is evident that chaos is strongly suppressed in the quantum case with respect to the classical case.

As $E\to\infty$ the quantum and classical models are equivalent, hence it is worth considering, in addition to the average measures of chaos $\mu(E,B)$ and $\langle\lambda\rangle(E,B)$, also an average measure of the relative importance of the quantum part of the Hamiltonian, given by the ensemble average $\langle Q\rangle(E,B)$, where for each single orbit Q is defined as

$$Q=\langle V_Q\rangle_t/\langle V\rangle_t . \tag{7}$$

Here $V_Q=(\hbar^2\sigma^2)/(8\xi^2)$ is the quantum correction to the potential, and V is the total potential—suitably normalized [15] in order that $0\leqslant Q\leqslant 1$—and $\langle\cdot\rangle_t$ stands for a time average along the orbit. The parameter $Q(E,B)$ has a smooth dependence on both E and B, at variance with the other parameters (Fig. 5). The transition from completely ordered, to mixed (ordered+chaotic), to almost completely chaotic dynamics that is observed in the quantum case as $B,E>0.3$ does not correspond to any transition from mainly quantum to mainly classical dynamics.

We now give a simple and intuitive explanation for the suppression of chaos by the quantum correction, in our model. Let us consider the map of the plane (ξ,η) obtained by a Poincaré section of the Hamiltonian system defined by Eq. (5), such as those reported in Figs. 2 and 3. As $B=0$ both the classical and the quantum system are integrable. Hence the trajectories of the map lie on invariant tori. Nevertheless, the geometry of such tori is dramatically different in the two cases: in the classical one, as $E>1/8$ a hyperbolic fixed point at $X=(0,0)$ exists, and the trajectory that passes

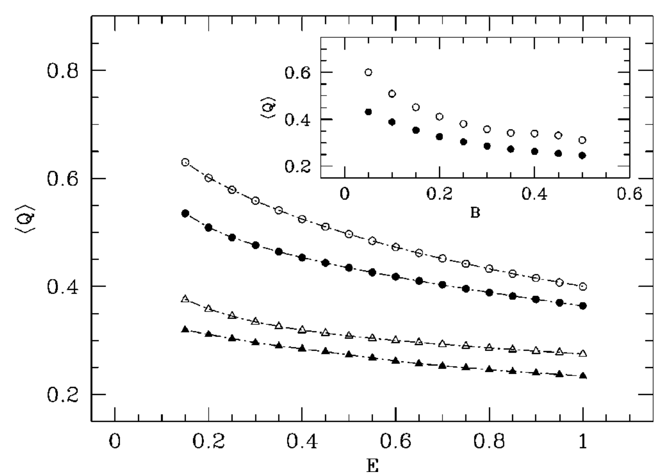

FIG. 5. Quantum parameter $\langle Q\rangle$ (see text) vs the energy E at different values of the field B. Symbols as in Fig. 1. Inset: $\langle Q\rangle$ vs the field B for two values of E, $E=0.2$ (solid circles) and $E=0.8$ (circles).

through X is actually a separatrix. Such a hyperbolic point is due to the presence of a local maximum in the potential. The quantum correction to the Hamiltonian is a ''centrifugal term'' that removes the local maximum of the potential and replaces it with an infinite barrier. Consequently, no hyperbolic fixed point in (0,0) exists in the quantum Poincaré section. As soon as the perturbation $-B\varphi$ is turned on, chaos immediately shows up in the classical case just in the neighborhood of X, because the stable and unstable manifolds, which constituted the separatrix of the unperturbed map, split and have infinite intersections [16]. In the quantum case this major source of chaos is removed because no separatrix exists in the unperturbed case, and chaos shows up only when the perturbation has completely distorted the original shape of the potential. In physical terms, the quantum suppression of classical chaos in our model is due to the fact that the quantum fluctuations are kept away from zero—consistently with Heisenberg's uncertainty principle—by a quantum term in the effective Hamiltonian.

To summarize, we have presented an example in which the phenomenon of quantum smoothing of classical chaos not only clearly shows up, but also finds a simple explanation on physical grounds. In order to understand how general this explanation can be, further work is needed. On one hand, the large N expansion belongs to semiclassical approximations, the validity of which, as far as chaos is concerned, can be questionable. In fact, in some cases it has been explicitly found that the onset of chaos is in correspondence to the breakdown of the approximation, and that the exact evolution of the quantum expectation values is not sensitive to the initial conditions [8,9]. Yet it has been argued that ''semiquantum chaos'' can be a real effect in open quantum systems since they are driven in a semiclassical regime by the interactions with the environment [8,17]. On the other hand, our results on quantum and classical Lyapunov exponents (see, e.g., Fig. 4) are in qualitative agreement with those reported in Ref. [6], where a quantum Lyapunov exponent—defined via the Bohm approach to quantum mechanics—was found positive but smaller than the classical one in a model of an hydrogen atom in an oscillating electric field. In Bohm's theory [18] particles obey classical equations of motion with an additional force derived from a ''quantum potential'' that is of order $\hbar^2$ as the quantum correction to the effective Hamiltonian (5) is. Bohm's equations of motion are exact, being completely equivalent to the standard quantum theory, but to write these equations for the system (1) would require the solution of the full time-dependent Schrödinger equation, thus the analysis of the exact Bohmian dynamics of our model system is practically unfeasible. Our treatment is approximate but tractable, and the features it shares with Bohmian mechanics are certainly suggestive.

We thank A. Barducci, G. Pettini, and M. Pettini for fruitful discussions and suggestions. L.C. acknowledges useful discussions with P. Castiglione and C. Presilla. This work has been carried out within the EEC program Human Capital and Mobility (Contracts Nos. OFES 950200, UE ERBCHRXCT 94-0579).

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